

# THIN POSITION FOR KNOTS IN A 3-MANIFOLD

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ABSTRACT. We extend the notion of thin multiple Heegaard splittings of a link in a 3-manifold to take into consideration not only compressing disks but also cut-disks for the Heegaard surfaces. We prove that if  $H$  is a  $c$ -strongly compressible bridge surface for a link  $K$  contained in a closed orientable irreducible 3-manifold  $M$  then one of the following is satisfied:

- $H$  is stabilized
- $H$  is meridionally stabilized
- $H$  is perturbed
- a component of  $K$  is removable
- $M$  contains an essential meridional surface.

## 1. INTRODUCTION

The notion of thin position for a closed orientable 3-manifold  $M$  was introduced by Scharlemann and Thompson in [8]. The idea is to build the 3-manifold by starting with a set of 0-handles, then alternate between attaching collections of 1-handles and 2-handles keeping the boundary at the intermediate steps as simple as possible and finally add 3-handles. Such a decomposition of a manifold is called a generalized Heegaard splitting. The classical Heegaard splitting where all 1-handles are attached at the same time followed by all 2-handles is an example of a generalized Heegaard splitting. Casson and Gordon [2] show that if  $A \cup_P B$  is a weakly reducible Heegaard splitting for  $M$  (i.e. there are meridional disks for  $A$  and  $B$  with disjoint boundaries), then either  $A \cup_P B$  is reducible or  $M$  contains an essential surface. Scharlemann and Thompson [8] show that such surfaces arise naturally when a Heegaard splitting is put in thin position.

Suppose a closed orientable 3-manifold  $M = A \cup_P B$  contains a link  $K$ , then we can isotope  $K$  so that it intersects each handlebody in boundary parallel arcs. In this case we say that  $P$  is a bridge surface for  $K$  or that  $P$  is a Heegaard surface for the pair  $(M, K)$ . The idea was first introduced by

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Schubert in the case that  $M = S^3$  and  $P = S^2$  and was extended by Morimoto and Sakuma for other 3-manifolds. In [5] Hayashi and Shimokawa considered multiple Heegaard splittings for  $(M, K)$  using the idea of changing the order in which the 1-handles and the 2-handles are attached. They generalized the result of [8] in this context, i.e. they showed that if  $P$  is a strongly compressible bridge surface for  $K$ , then either  $A \cup_P B$  is stabilized or cancellable or  $M - \eta(K)$  contains an essential meridional surface.

In this paper we will generalize this important result one step further by weakening the hypothesis. Suppose  $M$  is a compact orientable manifold and  $F \subset M$  is a properly embedded surface transverse to a 1-submanifold  $T \subset M$ . In some contexts it is necessary to consider not only compressing disks for  $F$  but also cut-disks, that is, disks whose boundary is essential on  $F - T$  and that intersect  $T$  in exactly one point, as for example in [1], [9] and [11]. A bridge surface  $P$  for a link  $K$  is  $c$ -strongly compressible if there is a pair of disjoint cut or compressing disks for  $P_K$  on opposite sides of  $P$ . In particular every strongly compressible bridge surface is  $c$ -strongly compressible. We will show that if a bridge surface  $P$  for  $K$  is  $c$ -strongly compressible then either it can be simplified in one of four geometrically obvious ways or  $(M, K)$  contains an essential meridional surface.

## 2. DEFINITIONS AND PRELIMINARIES

Let  $M$  be a compact orientable irreducible 3-manifold and let  $T$  be a 1-manifold properly embedded in  $M$ . A regular neighborhood of  $T$  will be denoted  $\eta(T)$ . If  $X$  is any subset of  $M$  we will use  $X_T$  to denote  $X - T$ . We will assume that any sphere in  $M$  intersects  $T$  in an even number of points. As all the results we will develop are used in the context when  $T$  only has closed components, this is a natural assumption. If  $K$  is a link in  $M$ , then any sphere in  $M$  intersects  $K$  in an even number of points, since the ball in bounds in  $M$  contains no endpoints of  $K$ .

Suppose  $F$  is a properly embedded surface in  $M$ . An *essential curve* on  $F_T$  is a curve that doesn't bound a disk on  $F_T$  and it is not parallel to a puncture of  $F_T$ . A *compressing disk*  $D$  for  $F_T$  is an embedded disk in  $M_T$  so that  $F \cap D = \partial D$  is an essential curve on  $F_T$ . A *cut-disk* is a disk  $D^c \subset M$  such that  $D^c \cap F = \partial D^c$  is an essential curve on  $F_T$  and  $|D \cap T| = 1$ . A *c-disk* is a cut or a compressing disk.  $F$  will be called *incompressible* if it has no compressing disks and *c-incompressible* if it has no  $c$ -disks.  $F$  will be called *essential* if it does not have compressing disks (it may have cut disks), it is not boundary parallel in  $M - \eta(T)$  and it is not a sphere that bounds a ball disjoint from  $T$ .

Suppose  $C$  is a compression body ( $\partial_- C$  may have some sphere components). A set of arcs  $t_i \subset C$  is *trivial* if there is a homeomorphism after

which each arc is either vertical, ie,  $t_i = (\text{point}) \times I \subset \partial_- C \times I$  or there is an embedded disk  $D_i$  such that  $\partial D_i = t_i \cup \alpha_i$  where  $\alpha_i \subset \partial_+ C$ . In the second case we say that  $t_i$  is  $\partial_+$ -parallel and the disk  $D_i$  is a bridge disk. If  $C$  is a handlebody, then all trivial arcs are  $\partial_+$ -parallel and are called *bridges*. If  $T$  is a 1-manifold properly embedded in a compression body  $C$  so that  $T$  is a collection of trivial arcs then we will denote the pair by  $(C, T)$ .

Let  $(C, T)$  be a pair of a compression body and a 1-manifold and let  $\mathcal{D}$  be the disjoint union of compressing disks for  $\partial_+ C$  together with one bridge disk for each  $\partial_+$ -parallel arc. If  $\mathcal{D}$  cuts  $(C, T)$  into a manifold homeomorphic to  $(\partial_- C \times I, \text{vertical arcs})$  together with some 3-balls, then  $\mathcal{D}$  is called a *complete disk system* for  $(C, T)$ . The presence of such a complete disk system can be taken as the definition of  $(C, T)$ .

Let  $M$  be a 3-manifold, let  $A \cup_P B$  be a Heegaard splitting (ie  $A$  and  $B$  are compression bodies) for  $M$  and let  $T$  be a 1-manifold in  $M$ . We say that  $T$  is in bridge position with respect to  $P$  if  $A$  and  $B$  intersect  $T$  only in trivial arcs. In this case we say that  $P$  is a bridge surface for  $T$  or that  $P$  as a Heegaard surface for the pair  $(M, T)$ .

Suppose  $M = A \cup_P B$  and  $T$  is in bridge position with respect to  $P$ . The Heegaard splitting is *c-strongly irreducible* if any pair of c-disks on opposite sides of  $P_T$  intersect, in this case the bridge surface  $P_T$  is *c-weakly incompressible*. If there are c-disks  $D_A \subset A$  and  $D_B \subset B$  such that  $D_A \cap D_B = \emptyset$ , the Heegaard splitting is *c-weakly reducible* and the bridge surface  $P_T$  is *c-strongly compressible*.

Following [5], the bridge surface  $P_T$  will be called *stabilized* if there is a pair of compressing disks on opposite sides of  $P_T$  that intersect in a single point. The bridge surface is *meridionally stabilized* if there is a cut disk and a compressing disk on opposite sides of  $P_T$  that intersect in a single point. Finally the bridge surface is called *cancellable* if there is a pair of canceling disks  $D_i$  for bridges  $t_i$  on opposite sides of  $P$  such that  $\emptyset \neq (\partial D_1 \cap \partial D_2) \subset (Q \cap K)$ . If  $|\partial D_1 \cap \partial D_2| = 1$  we will call the bridge surface *perturbed*. In [10] the authors show that if  $M = A \cup_P B$  is stabilized, meridionally stabilized or perturbed, then there is a simpler bridge surface  $P'$  for  $T$  such that  $P$  can be obtained from  $P'$  by one of three obvious geometric operations.

If the bridge surface  $P$  for  $T$  is cancellable with canceling disks  $D_1$  and  $D_2$  such that  $|\partial D_1 \cap \partial D_2| = 2$  then using this pair of disks some closed component  $t$  of  $T$  can be isotoped to lie in  $P$ . If this component can be isotoped to lie in the core of one of the compression bodies,  $A$  say, and is disjoint from all other bridge disks in  $A$  then  $A - \eta(t)$  is also a compression body and the 1-manifold  $T - t$  intersects it in a collection of trivial arcs. Thus  $(A - \eta(t)) \cup_P B$  is Heegaard splitting for  $(M - \eta(t))$  and  $P$  is a bridge surface for  $T - t$ . In this case we will say that  $T$  has a *removable*

*component*. A detailed discussion of links with removable components is given in [10].

In the absence of a knot, it follows by a theorem of Waldhausen that a Heegaard splitting of an irreducible manifold is stabilized if and only if there is a sphere that intersects the Heegaard surface in a single essential curve (i.e the Heegaard splitting is reducible), unless the Heegaard splitting is the standard genus 1 Heegaard splitting of  $S^3$ . We will say that a bridge surface for  $T$  is *c-reducible* if there is a sphere or a twice punctured sphere in  $M$  that intersects the bridge surface in a single essential closed curve. Then one direction of Waldhausen's result easily generalizes to bridge surfaces as the next theorem shows.

**Theorem 2.1.** *Suppose  $P$  is a bridge surface for a 1-manifold  $T$  properly embedded in a compact, orientable 3-manifold  $M$  where  $P$  is not the standard genus 1 Heegaard splitting for  $S^3$ . If  $P$  is stabilized, perturbed or meridionally stabilized then there exists a sphere  $S$ , possibly punctured by  $T$  twice, which intersects  $P$  in a single essential curve  $\alpha$  and neither component of  $S - \alpha$  is parallel to  $P$ .*

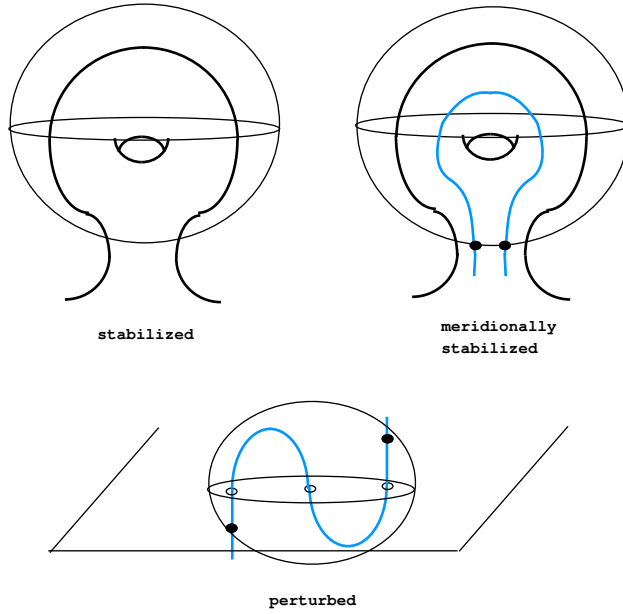


FIGURE 1.

*Proof.* If  $P$  is stabilized let  $S$  be the boundary of a regular neighborhood of the union of the pair of stabilizing disks, Figure 1. In this case  $S$  is a sphere disjoint from  $T$ . If  $P$  is meridionally stabilized, let  $S$  be the boundary of a

regular neighborhood of the union of the cut and compressing disks. In this case  $S$  is a twice punctured sphere with both punctures on the same side of  $S \cap P$ . Finally if  $P$  is perturbed with canceling disks  $E_1$  and  $E_2$ , let  $S$  be the boundary of a regular neighborhood of  $E_1 \cup E_2$ . Then  $S$  is a twice punctured sphere and the punctures are separated by  $S \cap P$ .  $\square$

### 3. C-COMPRESSION BODIES AND THEIR PROPERTIES

We will need to generalize the notion of a compression body containing trivial arcs as follows.

**Definition 3.1.** *A c-compression body  $(C, T)^c$  is a pair of a compression body  $C$  and a 1-manifold  $T$  such that there is a collection of disjoint bridge disks and c-disks  $\mathcal{D}^c$  so that  $\mathcal{D}^c$  cuts  $(C, T)^c$  into a 3-manifold homeomorphic to  $(\partial_- C \times I, \text{vertical arcs})$  together with some 3-balls. In this case  $\mathcal{D}^c$  is called a complete c-disk system.*

One way to construct a compression body is to take a product neighborhood  $F \times I$  of a closed, possibly disconnected, surface  $F$  so that any arc of  $T \cap (F \times I)$  can either be isotoped to be vertical with respect to the product structure or is parallel to an arc in  $F \times 0$  and then attach a collection of pairwise disjoint 2-handles  $\Delta$  to  $F \times 1$ . If we allow some of the 2-handles in  $\Delta$  to contain an arc  $t \subset T$  as their cocore, the resulting 3-manifold is a c-compression body. The complete c-disk system described in the definition above consists of all bridge disks together with the cores of the 2-handles. We will use this construction as an alternative definition of a c-compression body.

**Remark 3.2.** Recall that a spine of a compression body  $C$  is the union of  $\partial_- C$  together with a 1-dimensional graph  $\Gamma$  such that  $C$  retracts to  $\partial_- C \cup \Gamma$ . An equivalent definition of a c-compression body is that  $(C, T)^c$  is a compression body  $C$  together with a 1-manifold  $T$  and there exists a spine  $\Sigma$  for  $C$  such that all arcs of  $T$  that are not trivial in  $C$  can be simultaneously isotoped to lie on  $\Sigma$  and be pairwise disjoint. We will however not use this definition here.

**Proposition 3.3.** *Let  $(C, T)^c$  be a c-compression body. Then  $(C, T)^c$  is a compression body if and only if there is no arc  $t \subset T$  such that  $\partial t \subset \partial_- C$ . In particular if  $\partial_- C = \emptyset$ , then  $C$  is a handlebody.*

*Proof.* Consider the construction above and note that before the two handles are added no arc of  $T$  has both of its endpoints on  $F \times 1$ . If some 2-handle  $D$  attached to  $F \times 1$  contains an arc  $t \subset T$  as its core, this arc will have

both of its endpoints on  $\partial_- C$ . Thus  $C$  is a compression body if and only if no 2-handle contains such an arc.  $\square$

**Lemma 3.4.** *Let  $(C, T)^c$  be a  $c$ -compression body and let  $F$  be a  $c$ -incompressible,  $\partial$ -incompressible properly embedded surface transverse to  $T$ . Then there is a complete  $c$ -disk system  $\mathcal{D}^c$  of  $(C, T)^c$  such that  $\mathcal{D}^c \cap F$  consists of two types of arcs*

- *An intersection arc  $\alpha$  between a bridge disk in  $\mathcal{D}^c$  and a twice punctured sphere component of  $F$  with both endpoints of  $\alpha$  lying on  $T$ .*
- *An intersection arc  $\beta$  between a bridge disk in  $\mathcal{D}^c$  and a once-punctured disk component of  $F$  with one endpoint of  $\beta$  lying on  $T$  and the other lying on  $\partial_+ C$ .*

*Proof.* The argument is similar to the the proof of Lemma 2.2 in [5] so we only give an outline here. Let  $\mathcal{D}^c$  be a complete  $c$ -disk system for  $(C, T)^c$  chosen to minimize  $|\mathcal{D}^c \cap F|$ . Using the fact that  $F_T$  is  $c$ -incompressible, we may assume that  $\mathcal{D}^c \cap F$  does not contain any simple closed curves. If  $\alpha \subset \mathcal{D}^c \cap F$  is an arc with both of its endpoints on  $\partial C$ , then an outermost such arc either gives a  $\partial$ -compression for  $F$  contrary to the hypothesis or can be removed by an outermost arc argument contradicting the minimality of  $|\mathcal{D}^c \cap F|$ . Note that if  $\alpha$  lies on some cut-disk  $D^c$ , we can still choose the arc so that the disk it cuts from  $D^c$  does not contain a puncture. This establishes that  $F$  is disjoint from all  $c$ -disks in  $\mathcal{D}^c$ .

Suppose  $\alpha$  is an arc of intersection between a bridge disk  $E$  for  $T$  and a component  $F'$  of  $F$ . Assume that  $\alpha$  is an outermost such arc and let  $E' \subset E$  be the subdisk  $\alpha$  bounds on  $E$ . By the above argument at least one endpoint of  $\alpha$  must lie on  $T$ . If both endpoints of  $\alpha$  lie on  $T$ , the boundary of a regular neighborhood of  $E'$  gives a compressing disk for  $F$  contrary to the hypothesis unless  $F'$  is a twice punctured sphere. If  $\alpha$  has one endpoint on  $T$  and one endpoint on  $\partial C$ , a regular neighborhood of  $E'$  is a  $\partial$ -compressing disk for  $F$  unless  $F'$  is a once punctured disk.  $\square$

**Corollary 3.5.** *If  $(C, T)^c$  is a  $c$ -compression body, then  $\partial_- C$  is incompressible.*

*Proof.* Suppose  $D$  is a compressing disk for some component of  $\partial_- C$ . By Lemma 3.4, there exists a complete  $c$ -disk system  $\mathcal{D}^c$  for  $(C, T)^c$  such that  $D \cap \mathcal{D}^c = \emptyset$ . But this implies that  $D$  is a  $\partial$ -reducing disk for the manifold  $(F \times I, \text{vertical arcs})$ , a contradiction.  $\square$

If  $M$  is a 3-manifold we will denote by  $\tilde{M}$  the manifold obtained from  $M$  by filling any sphere boundary components of  $M$  with 3-balls.

**Lemma 3.6** (Lemma 2.4 [5]). *If  $F$  is an incompressible,  $\partial$ -incompressible surface in a compression body  $(C, T)$ , then  $F$  is a collection of the following kinds of components:*

- Spheres intersecting  $T$  in 0 or 2 points,
- Disks intersecting  $T$  in 0 or 1 points,
- Vertical annuli disjoint from  $T$ ,
- Closed surfaces parallel to a component of  $\partial_- \tilde{C}$ .

**Corollary 3.7.** *If  $F$  is a  $c$ -incompressible,  $\partial$ -incompressible surface in a  $c$ -compression body  $(C, T)^c$ , then  $F$  is a collection of the following kinds of components:*

- Spheres intersecting  $T$  in 0 or 2 points,
- Disks intersecting  $T$  in 0 or 1 points,
- Vertical annuli disjoint from  $T$ ,
- Closed surfaces parallel to a component of  $\partial_- \tilde{C}$ .

*Proof.* Delete all component of the first two types and let  $F'$  be the new surface. By Lemma 3.4, there exists a complete  $c$ -disk system  $\mathcal{D}$  for  $(C, T)^c$  such that  $\mathcal{D} \cap F' = \emptyset$ . Thus each component of  $F'$  is contained in a compression body with trivial arcs (in fact in a trivial compression body but we don't need this fact). The result follows by Lemma 3.6. □

#### 4. C-THIN POSITION FOR A PAIR 3-MANIFOLD, 1-MANIFOLD

The following definition was first introduced in [5]

**Definition 4.1.** *If  $T$  is a 1-manifold properly embedded in a compact 3-manifold  $M$ , we say that the disjoint union of surfaces  $\mathcal{H}$  is a multiple Heegaard splitting of  $(M, T)$  if*

- (1) *The closures of all components of  $M - \mathcal{H}$  are compression bodies  $(C_1, C_1 \cap T), \dots, (C_n, C_n \cap T)$ ,*
- (2) *for  $i = 1, \dots, n$ ,  $\partial_+ C_i$  is attached to some  $\partial_+ C_j$  where  $i \neq j$ ,*
- (3) *a component of  $\partial_- C_i$  is attached to some component of  $\partial_- C_j$  (possibly  $i = j$ ).*

*A component  $H$  of  $\mathcal{H}$  is said to be positive if  $H = \partial_+ C_i$  for some  $i$  and negative if  $H = \partial_- C_j$  for some  $j$ . The unions of all positive and all negative components of  $\mathcal{H}$  are denoted  $\mathcal{H}_+$  and  $\mathcal{H}_-$  respectively.*

Note that if  $\mathcal{H}$  has a single surface component  $P$ , then  $P$  is a bridge surface for  $T$ .

Using  $c$ -compression bodies instead of compression bodies, we generalize this definition as follows.

**Definition 4.2.** *If  $T$  is a 1-manifold properly embedded in a compact 3-manifold  $M$ , we say that the disjoint union of surfaces  $\mathcal{H}$  is a multiple  $c$ -Heegaard splitting of  $(M, T)$  if*

- (1) *The closures of all components of  $M - \mathcal{H}$  are  $c$ -compression bodies  $(C_1, C_1 \cap T)^c, \dots, (C_n, C_n \cap T)^c$ ,*
- (2) *for  $i = 1, \dots, n$ ,  $\partial_+ C_i$  is attached to some  $\partial_+ C_j$  where  $i \neq j$ ,*
- (3) *a component of  $\partial_- C_i$  is attached to some component of  $\partial_- C_j$  (possibly  $i = j$ )*

As in [8] and [5] we will associate to a multiple  $c$ -Heegaard splitting a measure of its complexity. The following notion of complexity of a surface is different from the one used in [5].

**Definition 4.3.** *Let  $S$  be a closed connected surfaces embedded in  $M$  transverse to a properly embedded 1-manifold  $T \subset M$ . The complexity of  $S$  is the ordered pair  $c(S) = (2 - \chi(S_T), g(S))$ . If  $S$  is not connected,  $c(S)$  is the multi-set of ordered pairs corresponding to each of the components of  $S$ .*

As in [8] the complexities of two possibly not connected surfaces are compared by first arranging the ordered pairs in each multi-set in non-increasing order and then comparing the two multi-sets lexicographically where the ordered pairs are also compared lexicographically.

**Lemma 4.4.** *Suppose  $S_T$  is meridional surface in  $(M, T)$  of non-positive euler characteristic. If  $S'_T$  is a component of the surface obtained from  $S_T$  by compressing along a  $c$ -disk, then  $c(S_T) > c(S'_T)$ .*

*Proof.* Without loss of generality we may assume that  $S_T$  is connected.

*Case 1:* Let  $\tilde{S}_T$  be a possibly disconnected surface obtained from  $S_T$  via a compression along a disk  $D$ . In this case  $\chi(S_T) < \chi(\tilde{S}_T)$  as  $\chi(D) = 1$  so the result follows immediately if  $\tilde{S}_T$  is connected. If  $\tilde{S}_T$  consists of two components then by the definition of compressing disk, we may assume that neither component is a sphere and thus both components of  $\tilde{S}_T$  have non-positive Euler characteristic. By the additivity of Euler characteristic it follows that if  $S'_T$  is a component of  $\tilde{S}_T$ , then  $\chi(\tilde{S}_T) \leq \chi(S'_T)$  so  $2 - \chi(S_T) > 2 - \chi(S'_T)$  as desired.

*Case 2:* Suppose  $\tilde{S}_T$  is obtained from  $S_T$  via a compression along a cut-disk  $D^c$ . If  $D^c$  is separating, then each of the two components of  $\tilde{S}_T$  has at least one puncture and if a component is a sphere, then it must have at least 3 punctures, ie each component of  $\tilde{S}_T$  has a strictly negative Euler characteristic. By the additivity of Euler characteristic, we conclude that for each component  $S'_T$  of  $\tilde{S}_T$ ,  $\chi(S'_T) < \chi(\tilde{S}_T) = \chi(S_T)$  and so the first component of the complexity tuple is decreased.



If the cut disk is not separating the cut-compression does not affect the first term in the complexity tuple as  $\chi(D^c) = 0$ . Note that  $\partial D^c$  must be essential in the non-punctured surface  $S$  so we can consider  $D^c$  as a compressing disk for  $S$  in  $M$ . Then  $g(\tilde{S}) < g(S)$  so in this case the second component of the complexity tuple is decreased.  $\square$

The *width* of a c-Heegaard splitting is the multiset of pairs  $w(\mathcal{H}) = c(\mathcal{H}_+)$ . In [5] a multiple Heegaard splitting is called *thin* if it is of minimum width amongst all possible multiple Heegaard splittings for the pair  $(M, T)$ . Similarly we will call a c-Heegaard splitting *c-thin* if it is of minimal width amongst all c-Heegaard splittings for  $(M, T)$ .

## 5. THINNING USING PAIRS OF DISJOINT C-DISKS

**Lemma 5.1.** *Suppose  $M$  is a compact orientable irreducible manifold and  $T$  is a properly embedded 1-submanifold. If  $P$  is a c-Heegaard splitting for  $(M, T)$  which is c-weakly reducible, then there exists a multiple c-Heegaard splitting  $\mathcal{H}'$  so that  $w(\mathcal{H}') < w(P)$ .*

*Moreover if  $M$  is closed then either*

- *There is a component of  $\mathcal{H}'_T$  that is neither an inessential sphere nor boundary parallel in  $M_T$ , or*
- *$P$  is stabilized, meridionally stabilized or perturbed, or a closed component of  $T$  is removable.*

The first part of the proof of this lemma is very similar to the proof of Lemma 2.3 in [5] and uses the idea of *untelescoping*. However, in Lemma 2.3 the authors only allow untelescoping using disks while we also allow untelescoping using cut-disks.

*Proof.* Let  $(A, A \cap T)^c$  and  $(B, B \cap T)^c$  be the two c-compression bodies that  $P$  cuts  $(M, T)$  into. Consider a maximal collection of c-disks  $\mathcal{D}_A^* \subset A_T$  and  $\mathcal{D}_B^* \subset B_T$  such that  $\partial \mathcal{D}_A^* \cap \partial \mathcal{D}_B^* = \emptyset$ . Let  $A'_T = cl(A_T - N(\mathcal{D}_A^*))$  and  $B'_T = cl(B_T - N(\mathcal{D}_B^*))$  where  $N(\mathcal{D}^*)$  is a collar of  $\mathcal{D}^*$ . Then  $A'_T$  and  $B'_T$  are each the disjoint union of c-compression bodies. Take a small collar  $N(\partial_+ A'_T)$  of  $\partial_+ A'_T$  and  $N(\partial_+ B'_T)$  of  $\partial_+ B'_T$ . Let  $C_T^1 = cl(A'_T - N(\partial_+ A'_T))$ ,  $C_T^2 = N(\partial_+ A'_T) \cup N(\mathcal{D}_B^*)$ ,  $C_T^3 = N(\partial_+ B'_T) \cup N(\mathcal{D}_A^*)$  and  $C_T^4 = cl(B'_T - N(\partial_+ B'_T))$ . This is a new multiple c-Heegaard splitting of  $(M, T)$  with positive surfaces  $\partial_+ C_1$  and  $\partial_+ C_2$  that can be obtained from  $P$  by c-compressing along  $\mathcal{D}_A^*$  and  $\mathcal{D}_B^*$  respectively and a negative surface  $\partial_- C_2 = \partial_- C_3$  obtained from  $P$  by compressing along both sets of c-disks. By Lemma 4.4 it follows that  $w(\mathcal{H}') < w(P)$ .

To show the second part of the lemma, suppose  $A \cup_P B$  is not stabilized, meridionally stabilized or perturbed and no component of  $T$  is removable

and, by way of contradiction, suppose that every component of  $\partial_- C_2$  is a sphere bounding a ball that intersects  $T$  in at most one trivial arc or a torus that bounds a solid torus such that  $t \subset T$  is a core curve of it.

Let  $\Lambda_A$  and  $\Lambda_B$  be the arcs that are the cocores of the collections of c-disks  $\mathcal{D}_A^*$  and  $\mathcal{D}_B^*$  respectively. If  $D^c$  is a cut-disk, we take  $\lambda \subset T$  as its cocore. Let  $\Lambda = \Lambda_A \cup \Lambda_B$  and note that  $P$  can be recovered from  $\partial_- C_3$  by surgery along  $\Lambda$ . As  $P$  is connected, at least one component of  $\partial_- C_3$  must be adjacent to both  $\Lambda_A$  and  $\Lambda_B$ , call this component  $F$ . Unless  $F_T$  is an inessential sphere or boundary parallel in  $M_T$  we are done. If  $F_T$  is an inessential sphere, then by Waldhausen's result the original Heegaard splitting is stabilized. As  $\partial M = \emptyset$  by hypothesis, the remaining possibility is that  $F_T$  is parallel in  $M_T$  to part of  $T$ ; since  $F_T$  is connected it is either a torus bounding a solid torus with a component of  $T$  as its core or  $F_T$  is an annulus, parallel to a subarc of  $T$ . That is  $F$  bounds a ball which  $T$  intersects in a trivial arc.

Let  $\mathcal{B}$  be the ball or solid torus  $F$  bounds. We will assume that  $\mathcal{B}$  lies on the side of  $F$  that is adjacent to  $\Lambda_A$  and that  $F$  is innermost in the sense that  $\mathcal{B} \cap \Lambda_B = \emptyset$ .

Let  $H = \partial_- C_3 \cap \mathcal{B}$  and let  $A'$  be the c-compression body obtained by adding the 1-handles corresponding to the arcs  $\Lambda_A \cap \mathcal{B}$  to a collar of  $H$ . (Some of these 1-handles might have subarcs of  $T$  as their core). Let  $B' = \mathcal{B} - A'$ . Notice that  $B'$  can be obtained from  $\mathcal{B}$  by c-compressing along all c-disks whose cocores are adjacent to  $F$  and thus  $B'$  is a c-compression body. In fact  $\partial_- B' = \emptyset$  so  $B'$  is a handlebody, let  $H' = \partial B'$ . Then  $A' \cup_{H'} B'$  is a c-Heegaard splitting for  $\mathcal{B}$  decomposing in into a c-compression body  $A'$  and a handlebody  $B'$ . There are two cases to consider:  $\mathcal{B}$  being a ball intersecting  $T$  in a trivial arc and  $\mathcal{B}$  being a torus. We will consider each case separately and prove that  $A' \cup_{H'} B'$  is actually a Heegaard splitting for  $\mathcal{B}$  (i.e.  $A'$  is a compression body) so we can apply previously known results.

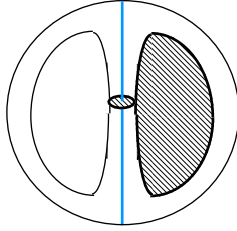


FIGURE 2.

**Case 1:** If  $\mathcal{B}$  is a ball and  $\mathcal{B} \cap T = t$  is a trivial arc, there are three sub-cases to consider. If  $t \cap H' \neq \emptyset$  then the construction above gives a nontrivial Heegaard splitting for the pair  $(\mathcal{B}, t)$ ;  $A'$  is a compression body

by Proposition 3.3 as  $\partial_- A'$  adjacent to two subarcs of  $t$  both of which have their second endpoint on  $\partial_+ A' = H'$ . By Lemma 2.1 of [4],  $H'$  is either stabilized or perturbed (in this context if  $H'$  is cancellable, it must be perturbed as  $t$  is not closed) so the same is true for  $P$ .

If  $t \subset A'$  and  $t = \Lambda \cap \mathcal{B}$  (in particular  $H = F$ ), Figure 2 shows a pair of c-disks demonstrating that  $P$  is meridionally stabilized.

If  $t \subset A'$  and  $t \neq \Lambda \cap \mathcal{B}$ , consider the solid torus  $V = \mathcal{B} - \eta(t)$ . Let  $A''$  be the c-compression body obtained by 1-surgery on  $H$  along the arcs  $\Lambda \cap V$ . As  $t \cap V = \emptyset$ ,  $A''$  is in fact a compression body. Note that  $V - A'' = B'$  as  $B' \cap t = \emptyset$ . Thus  $A'' \cup B'$  is a non-trivial Heegaard splitting for the solid torus  $V$ . By [7] it must be stabilized and thus so is  $P$ .

**Case 2:** Suppose  $F$  bounds a solid torus  $\mathcal{B}$ , which is a regular neighborhood of closed component  $t$  of  $T$ . As  $\partial_- A \cap t = \emptyset$ ,  $A' \cup_{H'} B'$  is a Heegaard splitting for  $(V, t)$ . By [3] it is cancellable or stabilized. This proves the theorem at hand unless  $H'$  is cancellable but not perturbed so assume this is the case. In particular this implies that  $H' \cap T = 2$ . In this case [3] shows that if  $g(H') \geq 2$  then  $H'$  is stabilized. Thus it remains to consider the case when  $H'$  is a torus intersecting  $t$  in two points. In this case  $H$  must be the union of  $F$  and a sphere  $S$  intersecting  $t$  in two points and  $\Lambda \cap \mathcal{B}$  is a single possibly knotted arc with one endpoint on  $F$  and the other on  $S$ . As  $t$  is cancellable, we can use the canceling disk in  $A'$  to isotope  $t$  across  $H'$  so it lies entirely in  $B'$ . After this isotopy it is clear that  $F$  and  $H'$  cobound a product region. As  $F$  is the boundary of a regular neighborhood of  $t$ , it follows that  $t$  is isotopic to the core loop of the solid torus  $B'$  ie,  $B' - \eta(t)$  is a trivial compression body.  $B$  can be recovered from  $B'$  by 1-surgery so  $B - \eta(t)$  is also a compression body. Thus after an isotopy of  $t$  along the pair of canceling disks,  $P$  is a Heegaard splitting for  $(M - \eta(t), T - t)$  so  $t$  is a removable component of  $T$ .

□

## 6. INTERSECTION BETWEEN A BOUNDARY REDUCING DISK AND A BRIDGE SURFACE

As in Jaco [6] a weak hierarchy for a compact orientable 2-manifold  $F$  is a sequence of pairs  $(F_0, \alpha_0), \dots, (F_n, \alpha_n)$  where  $F_0 = F$ ,  $\alpha_i$  is an essential curve on  $F_i$  and  $F_{i+1}$  is obtained from  $F_i$  by cutting  $F_i$  along  $\alpha_i$ . The final surface in the hierarchy,  $F_{n+1}$ , satisfies the following:

- (1) Each component of  $F_{n+1}$  is a disc or an annulus at least one boundary component of which is a component of  $\partial F$ .
- (2) Each non-annulus component of  $F$  has at least one boundary component which survives in  $\partial F_{n+1}$ .

The following lemma was first proven by Jaco and then extended in [5], Lemma 3.1.

**Lemma 6.1.** *Let  $F$  be a connected planar surface with  $b \geq 2$  boundary components. Let  $(F_0, \alpha_0), \dots, (F_n, \alpha_n)$  be a weak hierarchy with each  $\alpha_i$  an arc. If  $d$  is the number of boundary components of  $F_{n+1}$  then,*

- *If  $F_{n+1}$  does not have annulus components then  $d \leq b - 1$*
- *If  $F_{n+1}$  contains an annulus component, then  $d \leq b$ . If  $d = b$  and  $b \geq 3$ , then  $F_{n+1}$  contains a disc component.*

**Theorem 6.2.** *Suppose  $M$  is a compact orientable irreducible manifold and  $T$  is a properly embedded 1-manifold in  $M$ . Let  $A \cup_P B$  be a c-Heegaard splitting for  $(M, T)$ . If  $D$  is a boundary reducing disk for  $M$  then there exists such disk  $D'$  so that  $D'$  intersects  $P_T$  in a unique essential simple closed curve.*

*Proof.* Let  $D$  be a reducing disk for  $\partial M$  chosen amongst all such disks so that  $D \cap P$  is minimal. By Corollary 3.5,  $D \cap P \neq \emptyset$ . Let  $D_A = D \cap A$  and  $D_B = D \cap B$ .

Suppose some component of  $D_A$  is c-compressible in  $A$  with  $E$  the c-compressing disk. Let  $\gamma = \partial E$  and let  $D_\gamma$  be the disk  $\gamma$  bounds on  $D$ . Note that the sphere  $D_\gamma \cup E$  must be punctured by  $T$  either 0 or two times thus  $E$  must be a non-punctured disk. Let  $D' = (D - D_\gamma) \cup E$ .  $D'$  is also a reducing disk for  $\partial M$  as  $\partial D' = \partial D$  and  $D' \cap T = \emptyset$ . As  $\partial E$  is essential on  $D_A$ ,  $D_\gamma$  cannot lie entirely in  $A$  so  $|D_\gamma \cap P| > |E \cap P|$  and thus  $|D' \cap P| < |D \cap P|$  contradicting the choice of  $D$ . Similarly  $D_B$  is c-incompressible in  $B$ .

Suppose that  $E$  is a  $\partial$ -compressing disk for  $D_A$  and  $E$  is adjacent to  $\partial_- A$ .  $\partial$ -compressing  $D$  along  $E$  gives two disks  $D_1$  and  $D_2$  at least one of which has boundary essential of  $\partial M$ , say  $D_1$ . However  $|D_1 \cap P| < |D \cap P|$ , a contradiction.

Suppose that  $E$  is a  $\partial$ -compressing disk for  $D_A$  and  $E$  is adjacent to  $P$ . Let  $\alpha = E \cap D_A$ . Use  $E$  to isotope  $D$  so that a neighborhood of  $\alpha$  lies in  $B$ , call this new disk  $D^1$  and let  $D_A^1 = D^1 \cap A$  and  $D_B^1 = D^1 \cap B$ . Note that  $D_A^1$  is obtained from  $D_A$  by cutting along  $\alpha$  and  $D_A^1$  is also c-incompressible. Repeat the above operation naming each successive disk  $D^i$  until the resulting surface  $D_A^n = D^n \cap A$  is  $\partial$ -incompressible. By Corollary 3.7  $D_A^n$  consists of vertical annuli and disks.

Suppose some component of  $D_A$  is  $\partial$ -compressible but not adjacent to  $\partial_- A$ . In this case the result of maximally  $\partial$ -compressing this component has to be a collection of disks. By Case 1 of Lemma 6.1,  $|D_A^n \cap P| < |D_A \cap P|$  contradicting our choice of  $D$ . Thus every boundary compressible component of  $D_A$  is adjacent to  $\partial_- A$ , in particular  $\partial D \subset \partial_- A$  and  $D_A$  has a unique  $\partial$ -compressible component  $F$ . By the minimality assumption

and Case 2 of Lemma 6.1, some component of  $D_A^n$  must be a disk.  $D_B^n$  is then a planar surface that we have shown must be c-incompressible and has a component that is not a disk. As  $\partial D \cap \partial_- B = \emptyset$ , it follows that some component of  $D_B^n$  is  $\partial$ -compressible into  $P$  and disjoint from  $\partial_- B$ . The above argument applied to  $D_B^n$  leads to an isotopy of the disk  $D$  so as to reduce  $D \cap P$  contrary to our assumption. Thus  $D_A$  and  $D_B$  are both collections of vertical annuli and disks so  $D$  is a reducing disk for  $\partial M$  that intersects  $P$  in a single essential simple closed curve.  $\square$

**Corollary 6.3.** *Let  $A \cup_P B$  be a c-strongly irreducible c-Heegaard splitting of  $(M, T)$  and let  $F$  be a component of  $\partial M$ . If  $F_T$  is not parallel to  $P_T$ , then  $F_T$  is incompressible.*

*Proof.* Suppose  $D$  is a compressing disk for  $F_T \subset \partial_- A$  say. By Theorem 6.2 we can take  $D$  such that  $|D \cap P| = 1$ ,  $D_A = D \cap A$  is a compressing disk for  $P_T$  lying in  $A$  and  $D - D_B$  is a vertical annulus disjoint from  $T$ . As  $F_T$  is not parallel to  $P_T$ , there is a c-disk for  $P_T$  lying in  $A$ ,  $D_A$ . By a standard innermost disk and outermost arc arguments, we can take  $D_A$  so that  $D_A \cap D = \emptyset$ . But then  $D_A$  and  $D_B$  give a pair of c-weakly reducing disks for  $P_T$  contrary to our hypothesis.  $\square$

## 7. MAIN THEOREM

Following [5] we will call a c-Heegaard splitting  $\mathcal{H}$  *c-slim* if each component  $W_{ij} = C_i \cup C_j$  obtained by cutting  $M$  along  $\mathcal{H}_-$  is c-strongly irreducible and no proper subset of  $\mathcal{H}$  is also a multiple c-Heegaard splitting for  $M$ . Suppose  $\mathcal{H}$  is a c-thin c-Heegaard splitting of  $M$ . If some proper subset of  $\mathcal{H}$  is also a c-Heegaard splitting of  $M$ , then this c-Heegaard splitting will have lower width than  $\mathcal{H}$ . If some component  $W_{ij}$  of  $M - \mathcal{H}$  is c-weakly reducible, applying the untelescoping operation described in Lemma 5.1 to that component produces a c-Heegaard splitting of lower width. Thus if  $\mathcal{H}$  is c-thin, then it is also c-slim.

**Theorem 7.1.** *Suppose  $M$  is a closed orientable irreducible 3-manifold containing a link  $K$ . If  $P$  is a c-strongly compressible bridge surface for  $K$  then one of the following is satisfied:*

- $P$  is stabilized
- $P$  is meridionally stabilized
- $P$  is perturbed
- a component of  $K$  is removable
- $M$  contains an essential meridional surface  $F$  such that  $2 - \chi(F_K) \leq 2 - \chi(P_K)$ .

*Proof.* Let  $\mathcal{H}$  be a c-slim Heegaard splitting obtained from  $P$  by untelescoping as in Lemma 5.1, possibly in several steps. Let  $\mathcal{H}_-$  and  $\mathcal{H}_+$  denote the negative and positive surfaces of  $\mathcal{H}$  respectively and let  $W_{ij}$  be the components of  $M - \mathcal{H}_-$  where  $W_{ij}$  is the union of c-compression bodies  $C_i$  and  $C_j$  along  $H_{ij} = \partial_+ C_i = \partial_+ C_j$ . Suppose some component of  $\mathcal{H}_-$  is compressible with compressing disk  $D$ . By taking an innermost on  $D$  circle of  $D \cap \mathcal{H}_-$  we may assume that  $\partial_- C_i$  is compressible in  $W_{ij}$ . By Corollary 6.3 this contradicts our assumption that  $\mathcal{H}$  is c-slim. We conclude that  $\mathcal{H}_-$  is incompressible.

If some component of  $F_K$  of  $\mathcal{H}_-$  is neither an inessential sphere nor boundary parallel in  $M_K$ , then it is essential and  $2 - \chi(F_K) \leq 2 - \chi(P_K)$ . If every component is either an inessential sphere in  $M_K$  or boundary parallel, then by Lemma 5.1 the splitting is perturbed, stabilized, meridionally stabilized or there is a removable component.

□

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